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A CLASS OF GENERALIZED LÉVY LAPLACIANS IN INFINITE DIMENSIONAL CALCULUS

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ABSTRACT

A class of generalized Lévy Laplacians which contain as a special case the ordinary Lévy Laplacian are considered. Topics such as limit average of the second order functional derivative with respect to a certain equally dense (uniformly bounded) orthonormal base, the relations with Kuo's Fourier transform and other infinite dimensional Laplacians are studied.

1. Introduction and preliminaries

Hida's calculus which developed recently is a kind of infinite dimensional analogue of Schwartz's distribution theory based on the following Gel'fand triples,

$$(\mathcal{S}) \subset (L^2) \equiv L^2(\mathcal{S}(\mathbf{R}^1)^*, \mathcal{B}(\mathcal{S}(\mathbf{R}^1)^*), \mu) \subset (\mathcal{S})^*,$$

where $(\mathcal{S}(\mathbf{R}^1)^*, \mathcal{B}(\mathcal{S}(\mathbf{R}^1)^*), \mu)$ is the standard Gaussian space and (L^2) is a realization of Fock space on it. Infinite dimensional Laplacians similarly to the finite dimensional ones have been discussed within the framework of white noise

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calculus, i.e., Lévy Laplacian, Volterra Laplacian and Number operator [9], [10]. In particular, among these Laplacians Lévy Laplacian, by its essence of an infinite dimensional Laplacian, plays an important role in white noise calculus and its applications. It has been studied by many authors. It is particularly worth mentioning that T. Hida [4], motivated by Lévy's famous work over $H^2[0,1]$, introduced the so-called Lévy Laplacian Δ_L into the theory of generalized white noise functionals. An infinite dimensional rotation group characterization of Δ_L was obtained by N. Obata [16]. The relationship between Δ_L and Kuo's Fourier transform was also studied by T. Hida and K. Saitô in [8]. A class of Schrödinger equations which are closely associated with Δ_L were considered in [19]. On the other hand, recent interests aiming at its applications are, for instance, quantum mechanics, Feynman integral, quantum many time theory [6], [10], noncommutative quantum probability [11], [15] and vector-valued white noise functionals [17] and so on. The reader is referred to these papers for further details.

Recently, L. Accardi, P. Gibilisco and I. Volovich [1], motivated by the reformulation of Yang–Mills theory avoiding the uses of the "vector potential", proved that under some circumstances Yang–Mills equations are equivalent to certain Laplacian equations with respect to generalized Lévy Laplacians defined on the set of piecewise C^∞ functions ρ: $[0,1] \to \mathbf{R}^n$ for which $\rho(0)=0$. Therefore, from the white noise calculus point of view the following problems naturally arise: Do we have the similar generalized Lévy Laplacians and Laplacian equations in white noise calculus? What are the corresponding properties? In the present paper, we shall study the generalized Lévy Laplacians and their properties within the framework of white noise calculus.

Assume $\mathcal{A}$ denotes the self-adjoint extension of the following operator on $H = L^2(\mathbf{R}^1)$:

$$\mathcal{A}f(x) = \frac{d^2 f}{dx^2} + (1+x^2)f(x), \quad f \in C_0^\infty(\mathbf{R}^1).$$

This is called the harmonic oscillator operator [10]. By virtue of B. Simon [22], we can construct a class of Sobolev spaces over $\mathbf{R}^1$ by means of $\mathcal{A}$ as follows. Set

$$e_n(x) = (-1)^n(\pi^{1/2}2^n n!)^{-1/2} e^{x^2/2}\Big[\frac{d^n}{dx^n}e^{-x^2}\Big], \quad n \geq 0,$$

then $e_n(x) \in \mathcal{S}(\mathbf{R}^1)$ and we have $\mathcal{A}e_n = 2(n+1)e_n$, $\{e_n, n \geq 0\}$ is an ONB of $L^2(\mathbf{R}^1)$. Define for $p \geq 0$

$$\mathcal{S}_p(\mathbf{R}^1) \equiv \Big\{ f \in L^2(\mathbf{R}^1) : \|f\|_{2,p}^2 = \|\mathcal{A}^p f\|_2^2 = \sum_{n=0}^{\infty} < f, e_n > \big[2(n+1)\big]^{2p} < \infty \Big\}.$$

Then $\mathcal{S}_p(\mathbf{R}^1)$ is a Hilbert space and furthermore, for arbitrary $p \le q$, $\mathcal{S}_q(\mathbf{R}^1) \subset \mathcal{S}_p(\mathbf{R}^1)$ and

$$\mathcal{S}(\mathbf{R}^1) = \bigcap_{p \in \mathbf{R}_+} \mathcal{S}_p(\mathbf{R}^1), \quad \mathcal{S}(\mathbf{R}^1)^* = \bigcup_{p \in \mathbf{R}_+} \mathcal{S}_{-p}(\mathbf{R}^1).$$

Moreover, $\mathcal{S}_{-p}(\mathbf{R}^1)$ is the dual of $\mathcal{S}_p(\mathbf{R}^1)$. The dual pairing $< \cdot, \cdot >$ between $\mathcal{S}_{-p}(\mathbf{R}^1)$ and $\mathcal{S}_p(\mathbf{R}^1)$ is given by

$$\langle \phi, \psi \rangle = \sum_{n=0}^{\infty} \langle \phi, e_n \rangle \langle \psi, e_n \rangle, \quad \phi \in \mathcal{S}_{-p}(\mathbf{R}^1), \quad \psi \in \mathcal{S}_p(\mathbf{R}^1).$$

In other words, for arbitrary $p \ge 0$, $\mathcal{S}_p(\mathbf{R}^1)$ is the $L^2(\mathbf{R}^1)$-domain of $\mathcal{A}^p$ and $\|f\|_{2,p} = \|\mathcal{A}^p f\|_2$ where $\|\cdot\|_2$ is the norm of $L^2(\mathbf{R}^1)$. For $p \le 0$, we could also define the norm $\|\cdot\|_{2,p}$ similarly.

For $n \ge 2$, we put

$$\mathcal{S}_p(\mathbf{R}^n) =: \{ f \in L^2(\mathbf{R}^n) : \|f\|_{2,p} < \infty \}$$

where

$$\|f\|_{2,p}^2 =: \|\Gamma(\mathcal{A})^p f\|_2^2 = \sum_{k_1, \cdots, k_n} \prod_{i=1}^{n} \left[2(k_i + 1) \right]^{2p} \left| \langle f, e_{k_1} \hat{\otimes} \cdots \hat{\otimes} e_{k_n} \rangle \right|^2.$$

Here $\Gamma(\mathcal{A})$ denotes the second quantization operator of $\mathcal{A}$. $\Gamma(\mathcal{A})$ is defined on the dense subset spanned by $\{ f_1 \hat{\otimes} \cdots \hat{\otimes} f_n, f_i \in L^2(\mathbf{R}^1) \}$ of symmetric Fock space of H as follows:

$$\Gamma(\mathcal{A})(f_1 \hat{\otimes} \cdots \hat{\otimes} f_n) =: \mathcal{A} f_1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{A} f_n.$$

We denote simply by (L^2) the space $L^2(\mathcal{S}(\mathbf{R}^1)^*, \mathcal{B}(\mathcal{S}(\mathbf{R}^1)^*), \mu)$. Hence for each $\phi \in (L^2)$ there exists uniquely a sequence $\{ f^{(n)}, n \ge 1 \}$ in $\widehat{L^2(\mathbf{R}^n)}$ such that ϕ admits the Itô–Wiener decomposition

$$\phi = \sum_{n=0}^{\infty} I_n(f^{(n)})$$

where $I_n(f^{(n)})$ is the multiple Wiener integral of $f^{(n)}$ defined by

$$I_n(f^{(n)}) =: n! \int_{s_1 < \cdots < s_n} f^{(n)}(s_1, \ldots, s_n) dB_{s_1} \cdots dB_{s_n}.$$

As a direct consequence, we can easily have $\|\phi\|_2^2 = \sum_{n=0}^{\infty} n! \|f^{(n)}\|_2^2$.

In the sequel we usually write $\phi \sim (f^{(n)})$ to specify the sequence $\{f^{(n)}\}$. Now we are in a position to construct Sobolev spaces over the white noise space.

Suppose $p \geq 0$ and set

$$(\mathcal{S})_p =: \left\{\phi \in (L^2)\colon \phi \sim (f^{(n)}), \sum_{n=0}^{\infty} n! \|f^{(n)}\|_{2,p}^2 < \infty\right\};$$

then $(\mathcal{S})_p$ is a Hilbert space with the norm $\|\cdot\|_{2,p}$ given by

$$\|\phi\|_{2,p}^2 = \sum_{n=0}^{\infty} n! \|f^{(n)}\|_{2,p}^2 < \infty.$$

For each $p > 0$ we denote by $(\mathcal{S})_{-p}$ the dual space of $(\mathcal{S})_p$. Here we identify (L^2) with its dual. Each element of $(\mathcal{S})_{-p}$ corresponds uniquely to a sequence $\{f^{(n)}\}$ with $f^{(n)} \in \widehat{\mathcal{S}_{-p}(\mathbf{R}^n)}$ satisfying

$$\|\phi\|_{2,-p}^2 = \sum_{n=0}^{\infty} n! \|f^{(n)}\|_{2,-p}^2 < \infty.$$

For each $p > 0$ we have $(\mathcal{S})_p \subset (L^2) \subset (\mathcal{S})_{-p}$. $\{(\mathcal{S})_p, p \in \mathbf{R}^1\}$ is called Sobolev spaces over the white noise space. Define

$$(\mathcal{S}) = \bigcap_{p \in \mathbf{R}_+} (\mathcal{S})_p, \quad (\mathcal{S})^* = \bigcup_{p \in \mathbf{R}_+} (\mathcal{S})_{-p}.$$

We call the element of $(\mathcal{S})$ (resp. $(\mathcal{S})^*$) white noise test functional (resp. distribution).

The S-transform of functional $\phi \in (\mathcal{S})^*$ is defined by

$$(S\phi)(\xi) =: \langle\langle \phi, :e^{\langle \cdot, \xi \rangle}: \rangle\rangle, \quad \xi \in \mathcal{S}(\mathbf{R}^1)$$

where $:e^{\langle \cdot, \xi \rangle} :=: \exp\{\langle \cdot, \xi \rangle - 1/2 \cdot \|\xi\|_2^2\}$ in $(\mathcal{S})$ and $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the dual pairing between $(\mathcal{S})^*$ and $(\mathcal{S})$.

The Hida's differential operator ∂_t, $t \in \mathbf{R}^1$, is defined as

$$\partial_t \phi =: S^{-1}\left\{\frac{\delta}{\delta \xi(t)} (S\phi)(\xi)\right\}, \quad \xi \in \mathcal{S}(\mathbf{R}^1), \quad \phi \in (\mathcal{S})$$

where $\delta/\delta\xi(t)$ stands for Fréchet functional derivative [10]. This operator could also be interpreted as Gâteaux derivative in the direction of δ_t, delta function at t. More specifically, let $\phi \in (\mathcal{S})$ and x, $y \in \mathcal{S}(\mathbf{R}^1)^*$; the Gâteaux derivative of ϕ at x in the direction y is defined as

$$D_y \phi(x) = \frac{d}{ds} \phi(x + sy)\Big|_{s=0}.$$

It is well known that [10] for all $y \in \mathcal{S}(\mathbf{R}^1)^*$, D_y is a continuous linear operator on $(\mathcal{S})$ and, if $y \in \mathcal{S}(\mathbf{R}^1)$, it can actually be extended to a continuous linear operator on $(\mathcal{S})^*$. Accordingly, for all $y \in \mathcal{S}(\mathbf{R}^1)^*$ the dual operator D_y^* is a continuous linear operator on $(\mathcal{S})^*$ and, if $y \in \mathcal{S}(\mathbf{R}^1)$, its restriction is a continuous linear operator on $(\mathcal{S})$. For the special choice $y = \delta_t$, we have $\partial_t = D_{\delta_t}$.

Definition 1.1: (a) A white noise distribution ϕ is called an L-**functional** if for each $\xi \in \mathcal{S}(\mathbf{R}^1)$, there exist $(S\phi)'(\xi,\cdot) \in L^1_{loc}(\mathbf{R}^1)$, $(S\phi)_L(\xi,\cdot) \in L^1_{loc}(\mathbf{R}^1)$ and $(S\phi)_V(\xi,\cdot,\cdot) \in L^1_{loc}(\mathbf{R}^2)$ such that the first and the second order variations can be expressed uniquely in the forms

$$(S\phi)'(\xi)(\eta) = \int_{\mathbf{R}^1} (S\phi)'(\xi,t)\eta(t)dt,$$
$$(S\phi)''(\xi)(\eta,\zeta) = \int_{\mathbf{R}^2} (S\phi)_V(\xi,t,s)\eta(t)\zeta(s)dtds + \int_{\mathbf{R}^1} (S\phi)_L(\xi,t)\eta(t)\zeta(t)dt,$$

for each pair η, $\zeta \in \mathcal{S}(\mathbf{R}^1)$, and for any finite interval $T \subset \mathbf{R}^1$, $\int_T (S\phi)_L(\xi,t)dt$ is a U-functional.

(b) Let D_L denote the set of all L-functionals. For $\phi \in D_L$ and any finite interval $T \subset \mathbf{R}^1$, Lévy Laplacian Δ_L^T is defined by

$$\Delta_L^T \phi = S^{-1}\Big\{ \frac{1}{|T|} \int_T (S\phi)_L(\xi,t)dt \Big\}.$$

We also apply the same symbol Δ_L^T to the U-functional of the L-functional.

2. Generalized Lévy Laplacian

In this section we suggest a natural generalization of the usual Lévy Laplacian, originated by P. Lévy in $H = L^2[0,1]$ [14] as an infinite dimensional extension similarly to the finite dimensional Laplacian and subsequently were well studied in white noise calculus by a number of people mentioned above [10].

Our generalization differs from the original one (see [10]) which allows only δ-function type singular in the kernels of the second order functional derivative instead of permitting derivatives of any order of δ-function. The sort of higher order singularies have appeared in the reformulation of Yang–Mills theory in order to avoid the use of vector potential [1], [3].

Definition 2.1: Assume $\phi \in (\mathcal{S})^*$ and $F(\xi) = S\phi(\xi)$. For each $\xi \in \mathcal{S}(\mathbf{R}^1)$ we call ϕ or F an M-**order GLV (Generalized Lévy, Volterra)-functional** if $F(\xi)$

has the following finite functional derivatives of the forms: for η, $\zeta \in \mathcal{S}(\mathbf{R}^1)$

$$\langle F'(\xi),\eta\rangle = \sum_{i=0}^{M}\int_{\mathbf{R}^1} F_{(i)}(\xi,t)\eta^{(i)}(t)dt,$$

$$\langle F''(\xi),\eta\otimes\zeta\rangle = \sum_{i=0}^{M}\int_{\mathbf{R}^1} F_{GL,(i)}(\xi,t)\eta^{(i)}(t)\zeta^{(i)}(t)dt, + \sum_{i=0}^{M}\int_{\mathbf{R}^2} F_{GV,(i)}(\xi,s,t)\eta^{(i)}(s)\zeta^{(i)}(t)dsdt, \tag{2.1}$$

where $\eta^{(i)}(t)$ denotes the i-th order derivatives of $\eta(t)$ and $F_{(i)}$, $F_{GL,(i)}$ and $F_{GV,(i)}$ satisfy the following conditions:

(1) For each ξ, $F_{(i)}(\xi,\cdot) \in L^1_{loc}(\mathbf{R}^1)$, $F_{GL,(i)}(\xi,\cdot) \in L^1_{loc}(\mathbf{R}^1)$ and $F_{GV,(i)}(\xi,\cdot,\cdot) \in L^1_{loc}(\mathbf{R}^2)$ is symmetric.
(2) For any finite interval $T \subset \mathbf{R}^1$, $\int_T F_{GL,(i)}(\xi,t)dt < \infty$ and $\int_T F_{GV,(i)}(\xi,s,s)ds < \infty$ are both U-functionals.
(3) The integer M is taken as small as possible such that all conditions described above are satisfied and therefore the decomposition (2.1) is unique.

Bearing Definition 2.1 in mind, we are in a position to define the generalized Lévy Laplacian and Volterra Laplacian as follows.

Definition 2.2: For any M-order GLV functional $\phi \in (\mathcal{S})^*$, $F(\xi) = S\phi(\xi)$ and finite interval $T \subset \mathbf{R}^1$, $M \in \mathbf{N}$. The **generalized k-order ($k \le M$) Lévy Laplacian** $\Delta_{GL}^{(k)}\phi \in (\mathcal{S})^*$ (or $\Delta_{GL}^{(k)}F(\xi)$) is defined as

$$\Delta_{GL}^{(k)}\phi = S^{-1}\Big\{\frac{1}{|T|}\int_T F_{GL,(k)}(\xi,t)dt\Big\}$$

or

$$\Delta_{GL}^{(k)}F(\xi) = \frac{1}{|T|}\int_T F_{GL,(k)}(\xi,t)dt.$$

The **generalized k-order Volterra Laplacian** $\Delta_{GV}^{(k)}\phi \in (\mathcal{S})^*$ (or $\Delta_{GV}^{(k)}F(\xi)$) is defined as

$$\Delta_{GV}^{(k)}\phi = S^{-1}\Big\{\frac{1}{|T|}\int_T F_{GV,(k)}(\xi,t,t)dt\Big\}$$

or

$$\Delta_{GV}^{(k)}F(\xi) = \frac{1}{|T|}\int_T F_{GV,(k)}(\xi,t,t)dt.$$

Clearly, whenever $k = 0$ above, Definition 2.2 turns out to be the ordinary case. From now on, unless otherwise specified we suppose $k \neq 0$ in the following sections.

Example 2.1: Extended Normal Functionals. Assume

$$F(\xi)=\sum_{k=0}^{M}\int_{\mathbf{R}^n} f_k(t_1,\dots,t_n)(\xi^{(k)}(t_1))^{p_1}\cdots(\xi^{(k)}(t_n))^{p_n}dt_1\cdots dt_n$$

where $f_k \in L^1_{loc}(\mathbf{R}^n)$ and $\xi^{(k)}$ denotes the k-order derivative of $\xi(t)$.

By a direct computation, we have for arbitrary $\eta,\zeta\in\mathcal{S}(\mathbf{R}^1)$

$$\langle F'(\xi),\eta\rangle=\sum_{k=0}^{M}\sum_{j=1}^{n}p_j\int_{\mathbf{R}^n} f_k(t_1,\dots,t_n)(\xi^{(k)}(t_1))^{p_1}\cdots(\xi^{(k)}(t_j))^{p_j-1}\cdots(\xi^{(k)}(t_n))^{p_n}\cdot\eta^{(k)}(t_j)dt_1\cdots dt_n.$$

Therefore

$$\begin{aligned}\langle F''(\xi),\eta\otimes\zeta\rangle=&\sum_{k=0}^{M}\sum_{j=1}^{n}p_j(p_j-1)\int_{\mathbf{R}^n} f_k(t_1,\dots,t_n)\cdot(\xi^{(k)}(t_1))^{p_1}\\&\cdots(\xi^{(k)}(t_j))^{p_j-2}\cdots(\xi^{(k)}(t_n))^{p_n}\cdot\eta^{(k)}(t_j)\cdot\zeta^{(k)}(t_j)dt_1\cdots dt_n\\&+\sum_{k=0}^{M}\sum_{i,j=1}^{n}p_ip_j\int_{\mathbf{R}^n} f_k(t_1,\dots,t_n)(\xi^{(k)}(t_1))^{p_1}\cdots(\xi^{(k)}(t_i))^{p_i-1}\\&\cdots(\xi^{(k)}(t_j))^{p_j-1}\cdots(\xi^{(k)}(t_n))^{p_n}\cdot\eta^{(k)}(t_i)\cdot\zeta^{(k)}(t_j)dt_1\cdots dt_n.\end{aligned}$$

Thus, under some regular conditions we can suppose F is a GLV-functional and Definition 2.1 immediately yields that:

The k-order generalized Lévy part is

$$\sum_{j=1}^{n}p_j(p_j-1)\int_{\mathbf{R}^{n-1}} f_k(t_1,\dots,t_n)\cdot(\xi^{(k)}(t_1))^{p_1}\cdots(\xi^{(k)}(t_j))^{p_j-2}\cdots(\xi^{(k)}(t_n))^{p_n}dt_1\cdots\hat{dt_j}\cdots dt_n$$

and the k-order generalized Volterra part is

$$\sum_{i,j=1}^{n}p_ip_j\int_{\mathbf{R}^{n-2}} f_k(t_1,\dots,t_n)(\xi^{(k)}(t_1))^{p_1}\cdots(\xi^{(k)}(t_i))^{p_i-1}\cdots(\xi^{(k)}(t_j))^{p_j-1}\cdots(\xi^{(k)}(t_n))^{p_n}dt_1\cdots\hat{dt_i}\cdots\hat{dt_j}\cdots dt_n,$$

where the symbol "$\wedge$" denotes the term is deleted.

Example 2.2: Extended Exponential Functional. At the present moment, let us define $\phi =: \exp\{c\int_A f(t)(x^{(k)}(t))^2dt\}$: as a white noise distribution with its U-functional

$$F(\xi)=\exp\Big\{\frac{c}{1-2c}\int_A f(t)(\xi^{(k)}(t))^2dt\Big\}$$

where $f(t) \in L^1_{loc}(\mathbf{R}^1)$, $c \neq 1/2$, $A \subset \mathbf{R}^1$.

Clearly, letting $f(t) = 1$, $k = 0$ immediately yields the ordinary exponential functional

$$\phi =: \exp\Big(c\int_A x(t)^2 dt\Big) : .$$

On the other hand, by a direct computation we can easily obtain for any finite interval $T \subset \mathbf{R}^1$

$$\Delta^{(k)}_{GL}\phi = \frac{2c}{1-2c}\cdot\frac{\int_{T\cap A} f(t)dt}{|T|}\phi.$$

The next theorems show that under some circumstances, the generalized Lévy Laplacian $\Delta^{(k)}_{GL}$ is actually a derivative operator despite its definition being closely associated with a second order functional derivative.

THEOREM 2.1: *Suppose either (1) θ is a polynomially bounded C^2-function over $\mathbf{R}^1$ and F is an $\mathbf{R}^1$-valued M-order GLV-functional with smooth kernels, or (2) θ is a polynomially bounded analytic function on $\mathbf{C}$ and F is a $\mathbf{C}$-valued M-order GLV-functional with smooth kernels. Then $G(\xi) = \theta\cdot F(\xi)$ is also a GLV-functional and*

$$\Delta^{(k)}_{GL}G(\xi) = \theta'(F(\xi))\cdot\Delta^{(k)}_{GL}F(\xi),\quad \xi\in\mathcal{S}(\mathbf{R}^1),\quad k\le M.$$

Proof: First note that G is a U-functional since θ is polynomially bounded. On the other hand, for arbitrary ξ, η and ζ in $\mathcal{S}(\mathbf{R}^1)$, we have

$$\begin{aligned}
\langle G'(\xi),\eta\rangle =& \theta'(F(\xi))\cdot\langle F'(\xi),\eta\rangle\\
\langle G''(\xi),\eta\otimes\zeta\rangle =& \theta'(F(\xi))\langle F''(\xi),\eta\otimes\zeta\rangle + \theta''(F(\xi))\langle F'(\xi),\eta\rangle\langle F'(\xi),\zeta\rangle\\
=& \theta'(F(\xi))\Big\{\sum_{i=0}^{M}\int_{\mathbf{R}^2} F_{GV,(i)}(\xi,s,t)\eta^{(i)}(s)\zeta^{(i)}(t)dsdt\Big\}\\
&+ \theta'(F(\xi))\Big\{\sum_{j=0}^{M}\int_{\mathbf{R}^1} F_{GL,(j)}(\xi,t)\eta^{(j)}(t)\cdot\zeta^{(j)}(t)dt\Big\}\\
&+\sum_{i=0}^{M}\theta''(F(\xi))\int_{\mathbf{R}^1}\int_{\mathbf{R}^1}\tilde{F}_{(i)}(\xi,s,t))\eta^{(i)}(s)\zeta^{(i)}(t)dsdt
\end{aligned}$$

for some $\tilde{F}_{(i)}(\xi,\cdot,\cdot)\in L^1_{loc}(\mathbf{R}^2)$. Hence it is clear that

$$G_{GL,(k)}(\xi,t) = \theta'(F(\xi))\cdot F_{GL,(k)}(\xi,t),$$

which immediately implies that

$$\Delta^{(k)}_{GL}G(\xi) = \theta'(F(\xi))\cdot\Delta^{(k)}_{GL}F(\xi). \qquad \blacksquare$$

THEOREM 2.2: *Assume ϕ, $\psi \in (\mathcal{S})^*$ are both M-order GLV-functionals $F(\xi) = S\phi$, $G(\xi) = S\psi$, $H(\xi) = F(\xi)G(\xi)$. Then $\phi : \psi$ (Wick product of ϕ and ψ) is also the M-order GLV-functional and, for arbitrary $k \leq M$, we have*

$$\Delta_{GL}^{(k)}(\phi : \psi) = (\Delta_{GL}^{(k)}\phi) : \psi + \phi : (\Delta_{GL}^{(k)}\psi)$$

and

$$\Delta_{GL}^{(k)}H(\xi) = (\Delta_{GL}^{(k)}F(\xi)) \cdot G(\xi) + F(\xi) \cdot (\Delta_{GL}^{(k)}G(\xi)).$$

Proof: Assume $F(\xi)$, $G(\xi)$ have the following second order functional derivative decompositions. For any ξ, η and $\zeta \in \mathcal{S}(\mathbf{R}^1)$

$$\begin{aligned}\langle F''(\xi), \eta \otimes \zeta\rangle =& \sum_{i=0}^{M} \int_{\mathbf{R}^1} F_{GL,(i)}(\xi, t)\eta^{(i)}(t)\zeta^{(i)}(t)dt \\ &+ \sum_{i=0}^{M} \int_{\mathbf{R}^2} F_{GV,(i)}(\xi, s, t)\eta^{(i)}(s)\zeta^{(i)}(t)dsdt\end{aligned}$$

and

$$\begin{aligned}\langle G''(\xi), \eta \otimes \zeta\rangle =& \sum_{i=0}^{M} \int_{\mathbf{R}^1} G_{GL,(i)}(\xi, t)\eta^{(i)}(t)\zeta^{(i)}(t)dt \\ &+ \sum_{i=0}^{M} \int_{\mathbf{R}^2} G_{GV,(i)}(\xi, s, t)\eta^{(i)}(s)\zeta^{(i)}(t)dsdt.\end{aligned}$$

On the other hand, by a direct computation we easily have

$$\begin{aligned}\langle H''(\xi), \eta \otimes \zeta\rangle =& \langle F''(\xi), \eta \otimes \zeta\rangle G(\xi) + \langle G''(\xi), \eta \otimes \zeta\rangle F(\xi) \\ &+ \langle F'(\xi), \eta\rangle\langle G'(\xi), \zeta\rangle + \langle G'(\xi), \eta\rangle\langle F'(\xi), \zeta\rangle.\end{aligned}$$

Therefore, we can easily get

$$\Delta_{GL}^{(k)}H(\xi) = (\Delta_{GL}^{(k)}F(\xi)) \cdot G(\xi) + F(\xi) \cdot (\Delta_{GL}^{(k)}G(\xi)),$$

that is,

$$\Delta_{GL}^{(k)}(\phi : \psi) = (\Delta_{GL}^{(k)}\phi) : \psi + \phi : (\Delta_{GL}^{(k)}\psi). \qquad \blacksquare$$

We recall that the convolution of two Hida distributions ϕ and ψ is the white noise distribution $\phi * \psi$ defined by

$$\phi * \psi = \phi : \psi : g_{-2} \tag{2.2}$$

where g_c is a white noise distribution with its U-functional given as

$$(Sg_c)(\xi) = \exp\Big\{ -\frac{1}{2(c+1)}\|\xi\|_2^2\Big\}, \quad \xi \in \mathcal{S}(\mathbf{R}^1), \quad c \neq -1.$$

It is well known that

$$(2.3)\qquad (\Delta_L+1)(\phi*\psi)=\big((\Delta_L+1)\phi\big)*\psi+\phi*\big((\Delta_L+1)\psi\big).$$

Bearing (2.2) and (2.3) in mind, we are in a position to derive

COROLLARY 2.3: *Assume ϕ and ψ are in $\mathcal{D}(\Delta_{GL}^{(k)})$, the domain of Laplacian $\Delta_{GL}^{(k)}$; then $\phi:\psi$ is also in $\mathcal{D}(\Delta_{GL}^{(k)})$. Moreover, we have the following:*

$$\Delta_{GL}^{(k)}\phi*\psi=(\Delta_{GL}^{(k)}\phi)*\psi+\phi*(\Delta_{GL}^{(k)}\psi).$$

3. Relations with other Laplacians

In this section we shall consider the relations between generalized Lévy Laplacian $\Delta_{GL}^{(k)}$ and other infinite dimensional Laplacians N, Δ_G^*, $\Delta_{GV}^{(k)}$. The relation between $\Delta_{GL}^{(k)}$ and Δ_G will be considered in the next section.

Firstly, we recall that the Gross Laplacian is a white noise kernel operator given by

$$\Delta_G=\int\int_{\mathbf{R}^2}tr(s,t)\partial_s\cdot\partial_t dsdt$$

and the adjoint operator Δ_G^* of Δ_G is a white noise kernel operator given by

$$\Delta_G^*=\int\int_{\mathbf{R}^2}tr(s,t)\partial_s^*\cdot\partial_t^* dsdt.$$

Furthermore, for any ϕ in $(\mathcal{S})^*$ we have

$$S(\Delta_G^*\phi)(\xi)=\|\xi\|_2^2\cdot S\phi(\xi),\quad \xi\in\mathcal{S}(\mathbf{R}^1).$$

The reader is referred to [10] for further details on the aspect.

It is also well known that for each $\phi\in\mathcal{D}(\Delta_L)$,

$$\Delta_L\cdot\Delta_G^*\phi=2\phi+\Delta_G^*\cdot\Delta_L\phi.$$

THEOREM 3.1: *Assume $\phi\in\mathcal{D}(\Delta_{GL}^{(k)})$; then the following equality holds:*

$$\Delta_{GL}^{(k)}\cdot\Delta_G^*\phi=\Delta_G^*\cdot\Delta_{GL}^{(k)}\phi.$$

Proof: Set $F=S\phi$. Then the S-transform $H(\xi)$ of $\Delta_{GL}^*\phi$ is given by $H(\xi)=\|\xi\|_2^2F(\xi)$. Set $G(\xi)=\|\xi\|_2^2$ and we can easily see that $\Delta_{GL}^{(k)}G=0$. Hence, in terms of the derivativity of $\Delta_{GL}^{(k)}$, we have

$$\Delta_{GL}^{(k)}H(\xi)=\|\xi\|_2^2\cdot\left(\Delta_{GL}^{(k)}F(\xi)\right)$$

which immediately implies that

$$\Delta_{GL}^{(k)} \cdot \Delta_{GL}^{*}\phi = \Delta_{G}^{*} \cdot \Delta_{GL}^{(k)}\phi$$

as required. ∎

THEOREM 3.2: *Assume N is the number operator; then the following commutative relation holds:*

$$[N, \Delta_{GL}^{(k)}]\phi = N \cdot \Delta_{GL}^{(k)}\phi - \Delta_{GL}^{(k)} \cdot N\phi = -2\Delta_{GL}^{(k)}\phi$$

for all $\phi \in \mathcal{D}(\Delta_{GL}^{(k)})$.

Proof: Assume $\phi \in \mathcal{D}(\Delta_{GL}^{(k)})$, $F = S\phi$ and $G = S(N\phi)$. Then by the properties of N from [10], we have

$$G(\xi) = \langle F'(\xi), \xi \rangle.$$

Clearly,

$$\langle G''(\xi), \eta \otimes \zeta \rangle = 2\langle F''(\xi), \eta \otimes \zeta \rangle + \langle F'''(\xi), \xi \otimes \eta \otimes \zeta \rangle$$

which immediately implies that

$$G_{GL,(k)}(\xi, t) = 2F_{GL,(k)}(\xi, t) + \langle F_{GL,(k)}(\cdot, t)(\xi), \xi \rangle.$$

Therefore, we obtain that

$$S\Delta_{GL}^{(k)}(N\phi)(\xi) = 2S\Delta_{GL}^{(k)}\phi(\xi) + SN(\Delta_{GL}^{(k)}\phi)(\xi). \quad ∎$$

Suppose $\epsilon > 0$ and K_ϵ is a C^∞-function over $\mathbf{R}^1$ satisfying the following conditions:

(1) $\mathrm{supp}(K_\epsilon) \subset (-1/2, 1/2)$;
(2) $0 \leq K_\epsilon(t) \leq \epsilon^{-1}$ for all $t \in \mathbf{R}^1$;
(3) $\int_{-1/2}^{1/2} \phi(t)K_\epsilon(t)dt = 1$;
(4) $\lim_{\epsilon\to 0} \int_{-1/2}^{1/2} \phi(t)K_\epsilon(t)dt = \phi(0)$ for all $\phi \in C(\mathbf{R}^1)$;
(5) $\lim_{\epsilon\to 0} \epsilon\alpha(\epsilon) > 0$ where $\alpha(\epsilon) = \int_{-1/2}^{1/2} K_\epsilon(t)^2 dt$.

Recall that T is a fixed finite interval in $\mathbf{R}^1$ and define an integral operator Q_ϵ by

$$Q_\epsilon \xi(t) = \int_T K_\epsilon(t-s)\xi(s)ds.$$

Note that Q_ϵ is a Hilbert-Schmidt operator on $L^2(T)$ and is continuous from $L^2(T)$ into $\mathcal{S}(\mathbf{R}^1)$.

THEOREM 3.3: *Let T_1 denote the interval $\{x : \mathrm{dist}(x,T) \leq 1/2\}$. Assume F is a GLV-functional with the following decomposition given by*

$$\begin{aligned}\langle F''(\xi), \eta\otimes\zeta\rangle =&\sum_{i=0}^{M}\int_{\mathbf{R}^1} F_{GL,(i)}(\xi,t)\eta^{(i)}(t)\zeta^{(i)}(t)dt\\ &+\sum_{i=0}^{M}\int_{\mathbf{R}^2} F_{GV,(i)}(\xi,s,t)\eta^{(i)}(s)\zeta^{(i)}(t)dsdt.\end{aligned}$$

Assume that $F_{GL,(i)}(\xi,t)$ and $F_{GV,(i)}(\xi,s,t)$ are continuous in ξ, s and t. Suppose there exist nonnegative constants c, $c_{1,i}$, $c_{2,i}$, $c_{1,j,j}$, $c_{2,j,j} > 0$, i, $j \in \mathbf{N}$ and p, $q \geq 0$ such that

(a) $\sup\{|F_{GL,(i)}(\xi,t)|; t\in T_1\} \leq c_{1,i}\exp\left\{c_{2,i}\|\xi\|_{2,p}^2\right\}$ *for all ξ in $\mathcal{S}(\mathbf{R}^1)$;*

(b) $\sup\{|F_{GV,(j)}(\xi,s,t)|; s,t\in T_1\} \leq c_{1,j,j}\exp\left\{c_{2,j,j}\|\xi\|_{2,p}^2\right\}$ *for all ξ in $\mathcal{S}(\mathbf{R}^1)$;*

(c) $\|\mathcal{A}^p Q_\epsilon \mathcal{A}^{-q}\|_{L^2(\mathbf{R}^1)} \leq c$ *for all $\epsilon > 0$.*

Then the following equality holds:

$$\Delta_{GL}^{(k)}F = \lim_{\epsilon\to 0}\frac{1}{\alpha(\epsilon)}\Delta_{GV}^{(k)}(F\cdot Q_\epsilon).$$

Proof: Set $F_\epsilon = F\cdot Q_\epsilon$; then $F_\epsilon'' = Q_\epsilon \cdot F''(Q_\epsilon)\cdot Q_\epsilon$. Q_ϵ is a Hilbert-Schmidt operator defined by the kernel $K_\epsilon(t)$ and $Q_\epsilon\eta$ vanishes outside T_1 for arbitrary $\eta\in L^2(T)$. Thus for any η, $\zeta\in L^2(T)$, we have

$$\begin{aligned}\langle F_\epsilon''(\xi), \eta\otimes\zeta\rangle =&\langle F''(Q_\epsilon\eta), (Q_\epsilon\eta)\otimes(Q_\epsilon\zeta)\rangle\\ =&\sum_{i=0}^{M}\int_{T_1} F_{GL,(i)}(Q_\epsilon\xi;t)(Q_\epsilon\eta^{(i)}(t))(Q_\epsilon\zeta^{(i)}(t))dt\\ &+\sum_{i=0}^{M}\int\int_{T_1^2} F_{GV,(i)}(Q_\epsilon\xi;s,t)(Q_\epsilon\eta^{(i)}(s))(Q_\epsilon\zeta^{(i)}(t))dsdt.\end{aligned}\tag{3.1}$$

Using the properties of kernel function K_ϵ, we can easily rewrite (3.1) as follows:

$$\begin{aligned}\langle F_\epsilon''(\xi), \eta\otimes\zeta\rangle =&\sum_{i=0}^{M}\int\int_{T^2} A_{\epsilon,i}(\xi,s_1,s_2)\eta^{(i)}(s_1)\zeta^{(i)}(s_2)ds_1ds_2\\ &+\sum_{j=0}^{M}\int\int_{T^2} B_{\epsilon,j}(\xi,s_1,s_2)\eta^{(j)}(s_1)\zeta^{(j)}(s_2)ds_1ds_2\end{aligned}$$

where $A_{\epsilon,(i)}$ and $B_{\epsilon,(i)}$ are given by

$$\begin{aligned}A_{\epsilon,(i)}(\xi,s_1,s_2) &= \int_{T_1} F_{GL,(i)}(Q_\epsilon\xi,t)K_\epsilon(t-s_1)K_\epsilon(t-s_2)dt,\\ B_{\epsilon,(i)}(\xi,s_1,s_2) &= \int\int_{T^2} F_{GV,(i)}(Q_\epsilon\xi,s,t)K_\epsilon(s-s_1)K_\epsilon(t-s_2)dsdt.\end{aligned}$$

Hence we can easily obtain

$$\Delta_{GV}^{(k)} F_\epsilon(\xi) = \frac{1}{|T|}\Big\{ \int_T A_{\epsilon,(k)}(\xi, s, s)ds + \int_T B_{\epsilon,(k)}(\xi, s, s)ds\Big\}.$$

Now, by virtue of assumptions (b) and (c) we see that for all $\epsilon > 0$ and $\xi \in \mathcal{S}(\mathbf{R}^1)$,

$$\sup\{|F_{GV,(k)}(Q_\epsilon\xi, s, t)|, s, t \in T_1\} \le c_{1,k,k}\exp\Big\{c \cdot c_{2,k,k}\|\xi\|_{2,q}^2\Big\} \tag{3.2}$$

which, in addition to conditions (3) and (3.2), implies that

$$\begin{aligned}\int_T & |B_{\epsilon,(k)}(\xi, s, s)|ds \\ &\le c_{1,k,k}\exp\Big\{c \cdot c_{2,k,k}\|\xi\|_{2,q}^2\Big\} \int_T \Big[\int\int_{T^2} K_\epsilon(t_1 - s)K_\epsilon(t_2 - s)dt_1dt_2\Big]ds \\ &= c_{1,k,k}\exp\Big\{c \cdot c_{2,k,k}\|\xi\|_{2,q}^2\Big\} \int_T \Big[\int\int_{[-1/2,1/2]^2} K_\epsilon(u)K_\epsilon(v)dudv\Big]ds \\ &= c_{1,k,k}|T|\exp\Big\{c \cdot c_{2,k,k}\|\xi\|_{2,q}^2\Big\}.\end{aligned}$$

Recall that $\alpha(\epsilon) \to \infty$ as $\epsilon \to 0$, hence

$$\lim_{\epsilon\to 0} \frac{1}{\alpha(\epsilon)|T|} \int_T B_{\epsilon,(k)}(\xi, s, s)ds = 0.$$

On the other hand, in view of conditions (1) and (5) we can deduce

$$\begin{aligned}\frac{1}{\alpha(\epsilon)|T|} & \int_T A_{\epsilon,(k)}(\xi, s, s)ds - \frac{1}{|T|}\int_T F_{GL,(k)}(\xi, s)ds \\ &= \frac{1}{\alpha(\epsilon)|T|} \int_T \Big[A_{\epsilon,(k)}(\xi, s, s) - F_{GL,(k)}(\xi, s) \int_{T_1} K_\epsilon(t - s)^2dt\Big]ds \\ &= \frac{1}{\alpha(\epsilon)|T|} \int_T \Big[\int_{T_1} \Big(F_{GL,(k)}(Q_\epsilon\xi, t) - F_{GL,(k)}(\xi, s)\Big)K_\epsilon(t - s)^2dt\Big]ds.\end{aligned}$$

Hence, by virtue of condition (2) we deduce that

$$\begin{aligned}\Big|\frac{1}{\alpha(\epsilon)|T|} & \int_T A_{\epsilon,(k)}(\xi, s, s)ds - \frac{1}{|T|}\int_T F_{GL,(k)}(\xi, s)ds\Big| \\ &\le \frac{1}{\epsilon\alpha(\epsilon)|T|} \int_T \Big[\int_{T_1} \Big|F_{GL,(k)}(Q_\epsilon\xi, t) - F_{GL,(k)}(\xi, s)\Big|K_\epsilon(t - s)dt\Big]ds.\end{aligned}$$

On the other hand, by virtue of condition (3) we have

$$\begin{aligned}\int_T \Big\{\int_{T_1} & \Big|F_{GL,(k)}(Q_\epsilon\xi, t) - F_{GL,(k)}(\xi, t)\Big|K_\epsilon(t - s)dt\Big\}ds \\ &\le \int_{T_1} \Big|F_{GL,(k)}(Q_\epsilon\xi, t) - F_{GL,(k)}(\xi, t)\Big|dt.\end{aligned}$$

Therefore, by assumption (a) and dominated convergence theorem we have

$$\lim_{\epsilon\to 0}\int_T\left\{\int_{T_1}\left|F_{GL,(k)}(Q_\epsilon\xi,t)-F_{GL,(k)}(\xi,t)\right|K_\epsilon(t-s)dt\right\}ds=0.$$

Moreover, in terms of condition (1) we easily derive

$$\begin{aligned}\int_T\left\{\int_{T_1}\left|F_{GL,(k)}(Q_\epsilon\xi,t)-F_{GL,(k)}(\xi,t)\right|K_\epsilon(t-s)dt\right\}ds\\ =\int_T\left(\int_{T_1-s}\left|F_{GL,(k)}(\xi,u+s)-F_{GL,(k)}(\xi,s)\right|K_\epsilon(u)du\right)ds\\ =\int_T\left(\int_{-1/2}^{1/2}\left|F_{GL,(k)}(\xi,u+s)-F_{GL,(k)}(\xi,s)\right|K_\epsilon(u)du\right)ds\end{aligned}$$

which, in addition to condition (4), immediately implies that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_T\left\{\int_{T_1}\left|F_{GL,(k)}(\xi,t)-F_{GL,(k)}(\xi,s)\right|K_\epsilon(t-s)dt\right\}ds\\ =\int_T\left|F_{GL,(k)}(\xi,s)-F_{GL,(k)}(\xi,s)\right|ds=0.\end{aligned}$$

It follows from the results derived above that

$$\lim_{\epsilon\to 0}\frac{1}{\alpha(\epsilon)|T|}\int_T A_{\epsilon,(k)}(\xi,s,s)ds=\frac{1}{|T|}\int_T F_{GL,(k)}(\xi,s)ds.$$

The conclusion required now follows. ∎

4. Quasi-average of the generalized Lévy Laplacian

Recall that an orthonormal base $\{e_n; n\geq 1\}$ of $L^2(T)$ is called equally dense if the following equality holds, i.e., for any $f\in L^\infty(T)$, $T\subset \mathbf{R}^1$, a finite interval

$$\lim_{n\to 0}\frac{1}{n}\sum_{k=1}^{n}\int_T f(t)e_n^2(t)dt=\frac{1}{|T|}\int_T f(t)dt.$$

In particular, it is called uniformly bounded if $\sup\{\|e_n\|_\infty; n\geq 1\}\leq\infty$. We define quasi-average with respect to base $\{e_n\}$ as follows. For $k\in\mathbf{N}$,

$$E_kF(\xi)=:\lim_{n\to\infty}\frac{1}{n^k}\sum_{i=1}^{n}\langle F''(\xi)e_i,e_i\rangle$$

whenever the right side limit exists. Here $F(\xi)$ is supposed to be a U-functional.

It is well known that if $T=[0,1]$, (1) $\{1,\sqrt{2}\sin 2k\pi t,\sqrt{2}\cos 2k\pi t, k\geq 1\}$, (2) Walsh functions are examples of equally dense systems [5]. Throughout this section, we suppose $T=[0,1]$, $\{e_k\}=\{1,\sqrt{2}\sin 2k\pi t,\sqrt{2}\cos 2k\pi t, k\geq 1\}$.

THEOREM 4.1: *Assume F is a GLV-functional with the following second functional derivative decomposition: η, $\zeta \in \mathcal{S}(\mathbf{R}^1)$*

$$\langle F''(\xi), \eta \otimes \zeta \rangle = \sum_{i=0}^{k} \int_{\mathbf{R}^1} F_{GL,(i)}(\xi, s)\eta^{(i)}(s)\zeta^{(i)}(s)ds + \sum_{i=0}^{k} \int_{\mathbf{R}^2} F_{GV,(i)}(\xi, s, t)\eta^{(i)}(s)\zeta^{(i)}(t)dsdt.$$

Then we have

$$\Delta_{GL}^{(k)} F(\xi) = \frac{2k+1}{(2\pi)^{2k}} E_{2k+1} F(\xi), \quad k \in \mathbf{N}.$$

Proof: By the definition of quasi-average, we have for $e_k = \sqrt{2}\sin 2k\pi t$, $k \in \mathbf{N}$,

$$\begin{aligned}
&E_{2k+1}F(\xi)\\
&= \lim_{n\to\infty} \frac{1}{n^{2k+1}} \sum_{j=0}^{n} \langle F''(\xi)e_j, e_j\rangle\\
&= \lim_{n\to\infty} \frac{1}{n^{2k+1}} \sum_{j=0}^{n} 2\Big[\sum_{i_1=0}^{k} \int_0^1 \int_0^1 F_{GV,(i_1)}(\xi, s, t)(\sin 2\pi j s)^{(i_1)}(\sin 2\pi j t)^{(i_1)} dsdt\\
&\qquad + \sum_{j_1=0}^{k} \int_0^1 F_{GL,(j_1)}(\xi, t)(\sin 2\pi j t)^{(j_1)}(\sin 2\pi j t)^{(j_1)} dt\Big]\\
&= \lim_{n\to\infty} \frac{1}{n^{2k+1}} \sum_{j=1}^{n} 2\int_0^1 F_{GL,(k)}(\xi, t)(\sin 2\pi j t)^{(k)}(\sin 2\pi j t)^{(k)} dt\\
&= \lim_{n\to\infty} \frac{1}{n^{2k+1}} \sum_{j=1}^{n} \int_0^1 F_{GL,(k)}(\xi, t)(2\pi j)^{2k} f(j,t) dt\\
&= \frac{(2\pi)^{2k}}{2k+1} \int_0^1 F_{GL,(k)}(\xi, t)dt\\
&= \frac{(2\pi)^{2k}}{2k+1} \Delta_{GL}^{(k)} F
\end{aligned}$$

where $f(j,t) = \sqrt{2}\sin 2\pi j t$ or $\sqrt{2}\cos 2\pi j t$. ∎

THEOREM 4.2: *Let $\{1, \sqrt{2}\sin 2k\pi t, \sqrt{2}\cos 2k\pi t, k \geq 1\}$ be the equally dense and uniformly bounded orthonormal base of $L^2[0,1]$. Let P_n denote the projection onto the linear span of $\{e_1, \ldots, e_n\}$. Suppose $F \in \mathcal{D}(\Delta_{GL}^{(k)})$ and assume that, for arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|F_{GL,(k)}(\xi, t) - F_{GL,(k)}(\eta, t)| < \epsilon, \quad |\eta - \xi| < \delta.$$

Then we have

$$\Delta_{GL}^{(k)} = \frac{2k+1}{2\pi^{2k}} \lim_{n\to\infty} \frac{1}{n^{2k+1}} \Delta_G(F\cdot P_n).$$

Proof: Let $F_n = F\cdot P_n$; then $F_n''(\xi) = P_n F''(P_n\xi)P_n$. Hence $F_n''(\xi)$ is a trace class operator of $L^2([0,1])$ and, furthermore, $\Delta_G F_n$ is given by

$$\begin{aligned}\Delta_G F_n &= \sum_{k=1}^{\infty} \langle F''(P_n\xi), (P_n e_k)\otimes (P_n e_k)\rangle \\ &= \sum_{k=1}^{n} \langle F''(P_n\xi), e_k \otimes e_k\rangle.\end{aligned}$$

On the other hand, it is known that $F''(\xi)$ is given as follows:

$$\begin{aligned}\langle F''(\xi), \eta\otimes\zeta\rangle &= \sum_{i=0}^{k} \int_{\mathbf{R}^2} F_{GV,(i)}(\xi,s,t)\eta^{(i)}(s)\zeta^{(i)}(t)dsdt \\ &\quad + \sum_{i=0}^{k} \int_{\mathbf{R}^1} F_{GL,(i)}\eta^{(i)}(t)\zeta^{(i)}(t)dt.\end{aligned}$$

Thus, we have

$$\begin{aligned}\Delta_G F_n(\xi) &= \sum_{k=1}^{n} \int_0^1 F_L(P_n\xi,t)e_k^2(t)dt \\ &\quad + \sum_{k=1}^{n}\sum_{i=0}^{k} \int_0^1\int_0^1 F_{GV,(i)}(P_n\xi,s,t)e_k^{(i)}(s)e_k^{(i)}(t)dsdt \\ &\quad + \sum_{k=1}^{n}\sum_{i=1}^{k} \int_0^1 F_{GL,(i)}(P_n\xi,t)e_k^{(i)}(t)e_k^{(i)}(t)dt.\end{aligned}$$

Therefore, by the assumptions and carrying out the same arguments as in the proof of Theorem 4.1, we can prove that

$$\lim_{n\to\infty} \frac{1}{n^{2k+1}} \Delta_G F_n(\xi) = \frac{(2\pi)^{2k}}{2k+1} \Delta_{GL}^{(k)} F,$$

that is,

$$\Delta_{GL}^{(k)} F = \frac{2k+1}{(2\pi)^{2k}} \lim_{n\to\infty} \frac{1}{n^{2k+1}} \Delta_G(F\cdot P_n). \qquad \blacksquare$$

5. Relations with Kuo's Fourier transform

Let us first recall the definition of Kuo's Fourier transform.

Definition 5.1: The Fourier transform of a generalized white noise functional $\phi \in (\mathcal{S})^*$ is the unique generalized white noise functional, denoted by $\hat{\phi} \in (\mathcal{S})^*$, whose S-transform is given by

$$(S\hat{\phi})(\xi) = \langle\langle \bar{\phi}, e^{-i\langle \cdot, \xi\rangle} \rangle\rangle$$

or, equivalently,

$$(S\hat{\phi})(\xi) = (S\phi)(-i\xi) \cdot e^{-1/2\|\xi\|_2^2}, \quad \xi \in \mathcal{S}(\mathbf{R}^1).$$

It is worth pointing out that in [10] another more generalized infinite dimensional Fourier transform on $(\mathcal{S})^*$, Fourier–Mehler transform, is also considered. The Fourier–Mehler transform $F_\theta\phi$, $\theta \in \mathbf{R}^1$, $\phi \in (\mathcal{S})^*$, is defined as a unique generalized white noise functional with U-functional

$$S(F_\theta\phi)(\xi) = S\phi(e^{i\theta}\xi)\exp\{i/2 \cdot e^{i\theta}\sin(\|\xi\|_2^2)\}.$$

It is also well known that

$$\Delta_L\hat{\phi} + \widehat{\Delta_L\phi} = -\hat{\phi}.$$

However, for the generalized Lévy Laplacian we have the following

THEOREM 5.1: *Assume $\phi \in \mathcal{D}(\Delta_{GL}^{(k)})$, $k > 0$; then*

$$\Delta_{GL}^{(k)}\hat{\phi} + \widehat{\Delta_{GL}^{(k)}\phi} = 0$$

where "$\wedge$" in this section denotes Kuo's Fourier transform.

Proof: Let $\phi \in (\mathcal{S})^*$, $F = S\phi$ and $G = S\hat{\phi}$; then

$$G(\xi) = F(-i\xi) \cdot \exp\Big\{ - 1/2\|\xi\|_2^2\Big\},$$
$$\langle G'(\xi,t), \eta(t)\rangle = \langle -iF'(-i\xi,t)e^{-1/2\|\xi\|_2^2} - \xi(t)F(-i\xi)e^{-1/2\|\xi\|_2^2}, \eta(t)\rangle$$

and

$$
\begin{aligned}
\langle G''(\xi),&\eta\otimes\zeta\rangle \\
=&\langle -F''(-i\xi)(s,t)e^{-1/2\|\xi\|_2^2},\eta\otimes\zeta\rangle + ie^{-1/2\|\xi\|_2^2}\cdot\langle\xi,\eta\rangle \\
&\cdot\langle F'(-i\xi,t),\zeta(t)\rangle - F(-i\xi)e^{-1/2\|\xi\|_2^2}\langle\eta(t),\zeta(t)\rangle \\
&+ ie^{-1/2\|\xi\|_2^2}\langle\xi,\zeta\rangle\langle F'(-i\xi),\eta\rangle + e^{-1/2\|\xi\|_2^2}\cdot\langle\xi(t)\xi(t),\eta(t)\zeta(t)\rangle \\
=&-e^{-1/2\|\xi\|_2^2}\Big[\sum_{i=0}^{M}\int_{\mathbf{R}^2}F_{GV,(i)}(-i\xi,s,t)\eta^{(i)}(s)\zeta^{(i)}(t)dsdt \\
&+\sum_{i=0}^{M}\int_{\mathbf{R}^1}F_{GL,(i)}(-i\xi,t)\eta^{(i)}(t)\zeta^{(i)}(t)dt\Big] + ie^{-1/2\|\xi\|_2^2}\sum_{i=0}^{M}\int_{\mathbf{R}^2}\xi(t) \\
&\cdot F_{(i)}(-i\xi)(s)\eta(s)\zeta^{(i)}(t)dsdt - \int_{\mathbf{R}^1}F(-i\xi)e^{-1/2\|\xi\|_2^2}\eta(t)\zeta(t)dt \\
&+ ie^{-1/2\|\xi\|_2^2}\sum_{i=0}^{M}\int_{\mathbf{R}^2}\xi(t)F_{(i)}(-i\xi)(s)\eta^{(i)}(s)\zeta(t)dsdt \\
&+\int_{\mathbf{R}^2}e^{-1/2\|\xi\|_2^2}\xi(s)\xi(t)\eta(s)\zeta(t)dsdt.
\end{aligned}
$$

Consequently,

$$G_{GL,(k)}(\xi,t) = -e^{-1/2\|\xi\|_2^2}\cdot F_{GL,(k)}(-i\xi,t)$$

which immediately implies that

$$S(\Delta_{GL}^{(k)}\hat{\phi})(\xi) = -e^{-1/2\|\xi\|_2^2}\frac{1}{|T|}\int_T F_{GL,(k)}(-i\xi,t)dt. \tag{5.1}$$

On the other hand, we have

$$
\begin{aligned}
S(\widehat{\Delta_{GL}^{(k)}\phi})(\xi) =& S(\Delta_{GL}^{(k)})(-i\xi)\cdot e^{-1/2\|\xi\|_2^2} \\
=& e^{-1/2\|\xi\|_2^2}\frac{1}{|T|}\int_T F_{GL,(k)}(-i\xi,t)dt
\end{aligned}
$$

and

$$(S\hat{\phi})(\xi) = F(-i\xi)\cdot e^{-1/2\|\xi\|_2^2}$$

which, in addition to (5.1), immediately implies

$$\Delta_{GL}^{(k)}\hat{\phi} = -\widehat{\Delta_{GL}^{(k)}\phi}. \qquad \blacksquare$$

In a similar manner, we can also prove the following

THEOREM 5.2: *For any $\phi \in \mathcal{D}(\Delta_{GL}^{(k)})$ and any finite interval $T \subset \mathbf{R}^1$, the following equality holds:*

$$\Delta_{GL}^{(k)}(F_\theta\phi) = e^{2i\theta}F_\theta(\Delta_{GL}^{(k)}\phi)$$

where $F_\theta\phi$ is the Fourier–Mehler transform of ϕ, $\theta \in \mathbf{R}^1$.

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References

[1] L. Accardi, L. Gibilisco and I. V. Volovich, *The Lévy Laplacian and Yang–Mills equation,* Preprint of Vito Volterra Centre, Dec. 92N. 129.

[2] I. Ya. Aref'eva and I. Volovich, *Higher order functional conservation laws in gauge theories,* in *Generalized Functions and Their Applications in Mathematical Physics*, Proceedings of the International Conference, Computation Centre of the Academy of Science of the USSR, Moscow, 1981.

[3] L. Gross, *A Poincaré lemma for connection form,* Journal of Functional Analysis **63** (1985), 1–48.

[4] T. Hida, *Analysis of Brownian functionals*, Carleton-Ottawa Mathematical Lecture Notes Series No. 13, Carleton University, Ottawa, 1975.

[5] T. Hida, *Brownian Motion,* Springer-Verlag, New York, Berlin, 1980.

[6] T. Hida, *Random fields as generalized white noise functionals*, Proc. IFIP, Enschede, 1992.

[7] T. Hida and K. Liu, *On the generalized Volterra Laplacian in white noise calculus,* Preprint, 1998.

[8] T. Hida and K. Saitô, *White noise analysis and the Lévy Laplacian,* Stochastic Processes in Physics and Engineering (1988), 177–184.

[9] T. Hida, N. Obata and K. Saitô, *Infinite dimensional rotations and Laplacians in terms of white noise calculus,* Nagoya Mathematical Journal **128** (1992), 65–93.

[10] T. Hida, H-H Kuo, J. Potthoff and K. Streit, *White Noise: An Infinite Dimensional Calculus,* Kluwer Academic Publishers, Boston, 1993.

[11] Z. H. Huang, *Quantum white noise — white noise approach to quantum stochastic calculus,* Nagoya Mathematical Journal **129** (1993), 23–43.

[12] H. H Kuo, *On the Laplacian operators of generalized Brownian functionls*, in *Stochastic Processes and their Applications* (K. Itô and T. Hida, eds.), Lecture Notes in Mathematics **1203**, 1986, pp. 119–128.

[13] H. H Kuo, N. Obata and K. Saitô, *Lévy Laplacian of generalized functions on a nuclear space*, Journal of Functional Analysis **94** (1990), 74–92.

[14] P. Lévy, *Problémes concrets d'analyse fonctionnelle*, Gauthier-Villars, Paris, 1951.

[15] Y. L. Liao and K. Liu, *On the Fermion quantum white noise calculus*, Journal of Physics. A. Mathematical and General **26** (1993), 6421–6430.

[16] N. Obata, *A characterization of the Lévy Laplacian in terms of infinite dimensional rotation groups*, Nagoya Mathematical Journal **118** (1990), 111–132.

[17] N. Obata, *Operator calculus on vector-valued white noise functionals*, Journal of Functional Analysis **121** (1994), 185–232.

[18] J. Potthoff and L. Streit, *A characterization of Hida distributions*, Journal of Functional Analysis **101** (1991), 212–229.

[19] K. Saitô, *Itô's formula and Lévy Laplacian. (I), (II)*, Nagoya Mathematical Journal **108** (1987), 67–76; **116** (1991), 153–169.

[20] K. Saitô, *A (C_0)-group generated by the Lévy Laplacian*, Stochastic Analysis and Applications **16** (1998), 567–584.

[21] L. Schwartz, *Theorie des Distributions*, Hermann, Paris, 1950.

[22] B. Simon, *Distributions and Hermite expansion*, Journal of Mathematical Physics **12** (1971), 140–148.

[23] B. Simon, *The $P(\phi_2)$ Euclidean (Quantum) Fields Theory*, Princeton University Press, 1974.

[24] T. T. Wu and C. N. Yang, *Some remarks about unquantized non-Abelian gauge fields*, Physical Review D **12** (1975), 3843–3844.